Scientific Computing — Point Processes

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Neuroethology

WS 14/15

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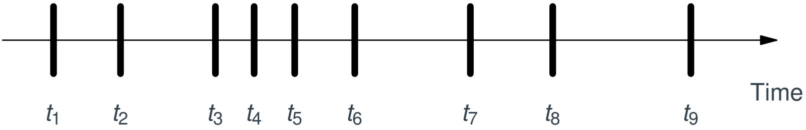
Content

Point process

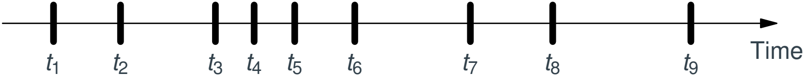


A point process is a stochastic (or random) process that generates a sequence of events at times $\{t_i\}$, $t_i \in \mathbb{R}$.
For each point process there is an underlying continuous-valued process evolving in time. The associated point process occurs when the underlying continuous process crosses a threshold. Examples:

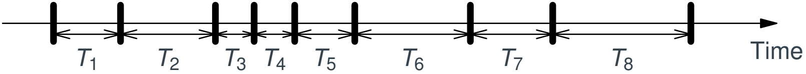
- Spikes/heartbeat: generated by the dynamics of the membrane potential of neurons/heart cells.
- Earth quakes: generated by the pressure dynamics between the tectonic plates on either side of a geological fault line.
- Onset of cricket/frogs/birds/... songs: generated by the dynamics of the state of a nervous system.

Point process

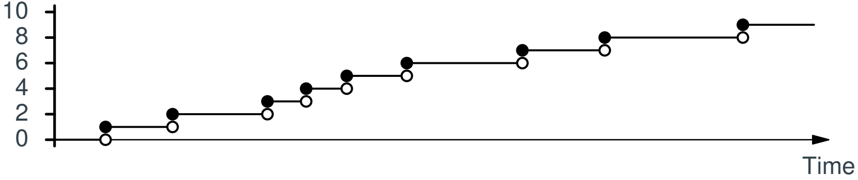
Event times $\{t_i\}$



Intervals $\{T_i\}$, $T_i = t_{i+1} - t_i$



Event counts $\{n_i\}$

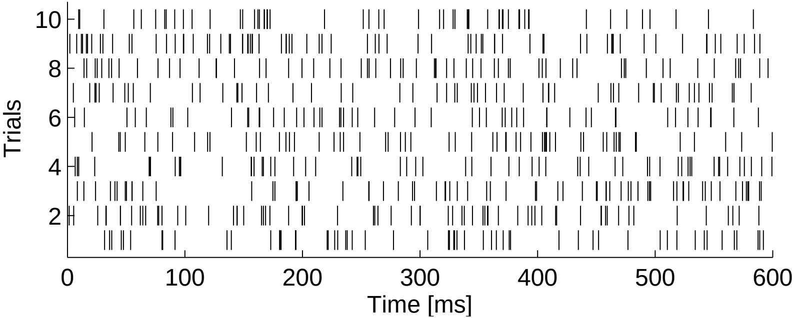


Homogeneous Poisson process

The probability $p(t)\delta t$ of an event occuring at time t is independent of t and independent of any previous event (independent of event history).
The probability P for an event occuring within a time bin of width Δt is

$$P = \lambda \cdot \Delta t$$

for a Poisson process with rate λ.



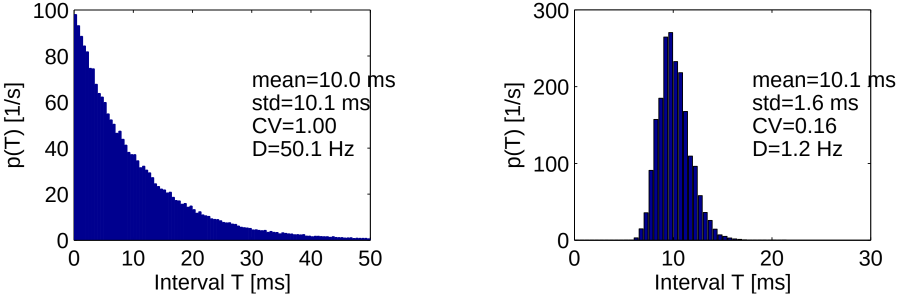
Rate

Rate of events r (“spikes per time”) measured in Hertz.

- Number of events N per observation time W: $r = \frac{N}{W}$
- Without boundary effects: $r = \frac{N-1}{t_N - t_1}$
- Inverse interval: $r = \frac{1}{\mu_{ISI}}$

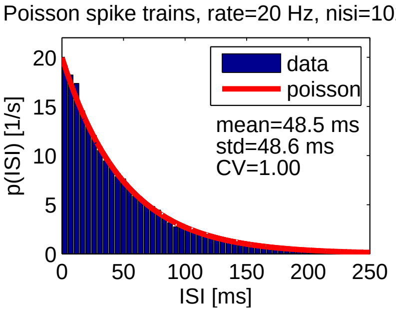
(Interspike) interval statistics

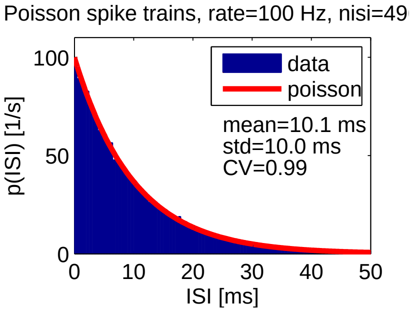
- Histogram $p(T)$ of intervals T. Normalized to $\int_0^\infty p(T)\, dT = 1$
- Mean interval $\mu_{ISI} = \langle T \rangle = \frac{1}{n} \sum_{i=1}^{n} T_i$
- Variance of intervals $\sigma^2_{ISI} = \langle (T - \langle T \rangle)^2 \rangle$
- Coefficient of variation $CV_{ISI} = \frac{\sigma_{ISI}}{\mu_{ISI}}$
- Diffusion coefficient $D_{ISI} = \frac{\sigma^2_{ISI}}{2\mu^3_{ISI}}$



Interval statistics of homogeneous Poisson process

- Exponential distribution of intervals T: $p(T) = \lambda e^{-\lambda T}$
- Mean interval $\mu_{ISI} = \frac{1}{\lambda}$
- Variance of intervals $\sigma^2_{ISI} = \frac{1}{\lambda^2}$
- Coefficient of variation $CV_{ISI} = 1$

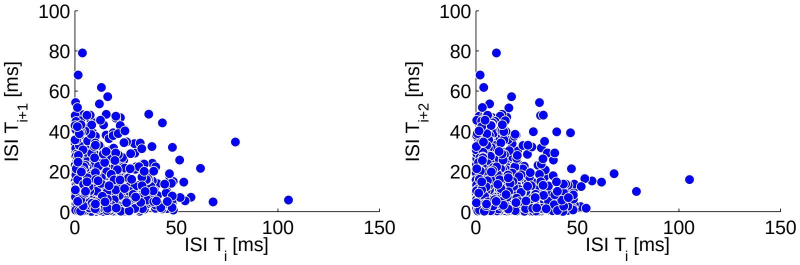




Interval return maps

Scatter plot between succeeding intervals separated by lag k.

Poisson process $\lambda = 100\,\text{Hz}$:

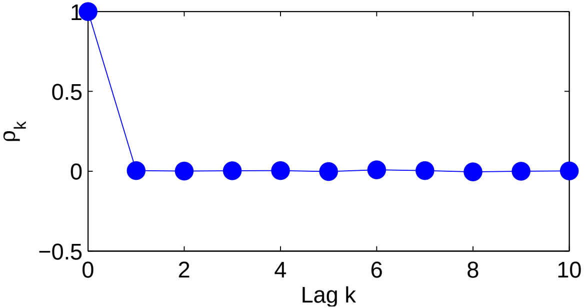


Serial interval correlations

Correlation coefficients between succeeding intervals separated by lag k:

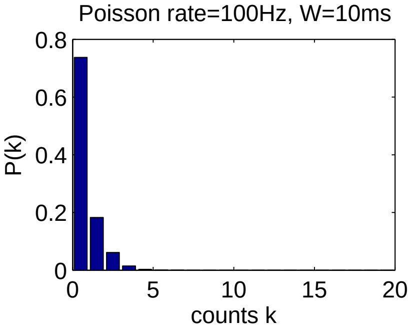
$$\rho_k = \frac{\langle (T_{i+k} - \langle T \rangle)(T_i - \langle T \rangle) \rangle}{\langle (T_i - \langle T \rangle)^2 \rangle} = \frac{\mathrm{cov}(T_{i+k}, T_i)}{\mathrm{var}(T_i)}$$

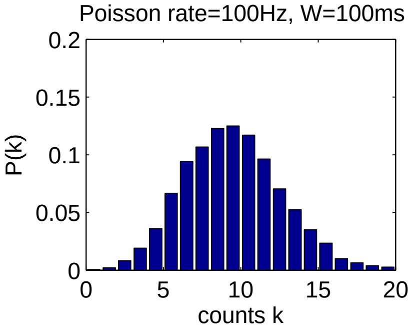
- $\rho_0 = 1$ (correlation of each interval with itself).
- Poisson process: $\rho_k = 0$ for $k > 0$ (renewal process!)



Count statistics

Histogram of number of events *N* (counts) within observation window of duration *W*.

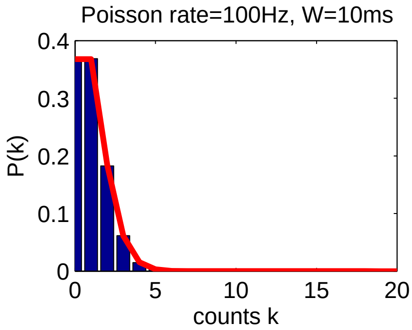


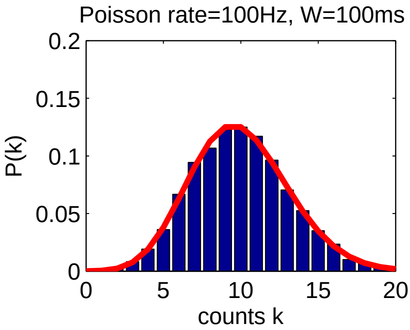


Count statistics of Poisson process

Poisson distribution:

$$P(k) = \frac{(\lambda W)^k e^{\lambda W}}{k!}$$



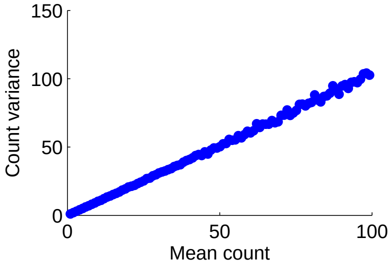


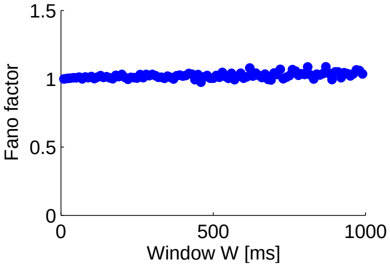
Count statistics — Fano factor

Statistics of number of events N within observation window of duration W.

- Mean count: $\mu_N = \langle N \rangle$
- Count variance: $\sigma_N^2 = \langle (N - \langle N \rangle)^2 \rangle$
- Fano factor (variance divided by mean): $F = \frac{\sigma_N^2}{\mu_N}$
- Poisson process: $F = 1$

Poisson process $\lambda = 100\,\text{Hz}$:





Integrate-and-fire models

Leaky integrate-and-fire model (LIF):

$$\tau \frac{dV}{dt} = -V + RI + D\xi$$

Whenever membrane potential $V(t)$ crosses the firing threshold θ, a spike is emitted and $V(t)$ is reset to V_{reset}.

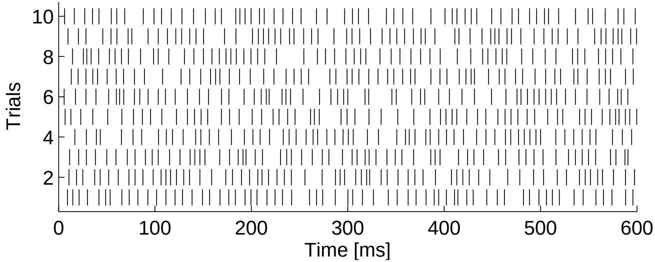
- τ: membrane time constant (typically 10 ms)
- R: input resistance (here 1 mV (!))
- $D\xi$: additive Gaussian white noise of strength D
- θ: firing threshold (here 10 mV)
- V_{reset}: reset potential (here 0 mV)

Integrate-and-fire models

Discretization with time step Δt: $V(t) \rightarrow V_i,\ t_i = i\Delta t$.
Euler integration:

$$\begin{aligned} \frac{dV}{dt} &\approx \frac{V_{i+1} - V_i}{\Delta t} \\ \Rightarrow \quad V_{i+1} &= V_i + \Delta t \frac{-V_i + RI_i + \sqrt{2D\Delta t} N_i}{\tau} \end{aligned}$$

N_i are normally distributed random numbers (Gaussian with zero mean and unit variance) — the $\sqrt{\Delta t}$ is for white noise.

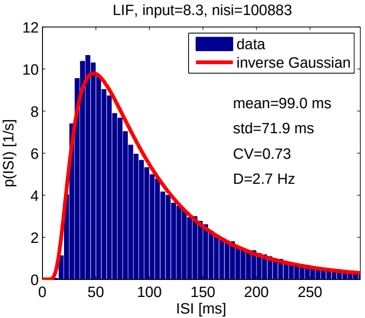


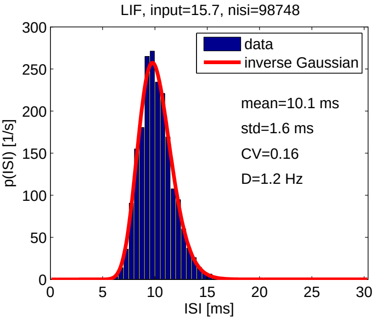
Interval statistics of LIF

Interval distribution approaches Inverse Gaussian for large I:

$$p(T) = \frac{1}{\sqrt{4\pi D T^3}} \exp\left[-\frac{(T - \langle T \rangle)^2}{4DT\langle T \rangle^2}\right]$$

where $\langle T \rangle$ is the mean interspike interval and D is the diffusion coefficient.



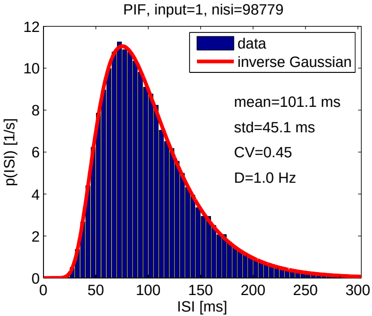


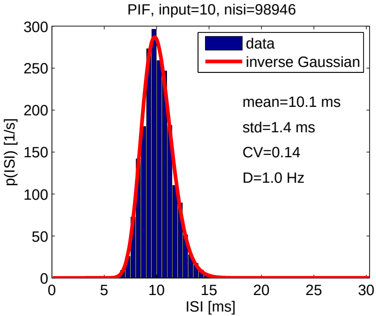
Interval statistics of PIF

For the perfect integrate-and-fire (PIF)

$$\tau \frac{dV}{dt} = RI + D\xi$$

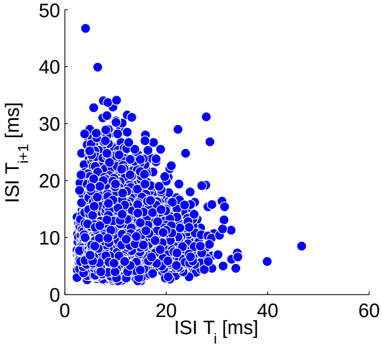
(the canonical model or supra-threshold firing on a limit cycle)
the Inverse Gaussian describes exactly the interspike interval distribution.

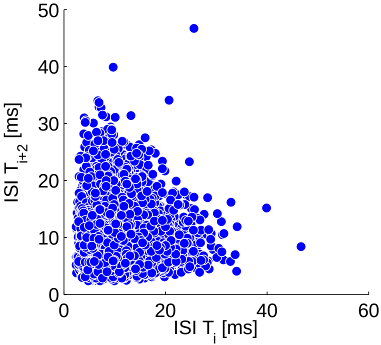




Interval return map of LIF

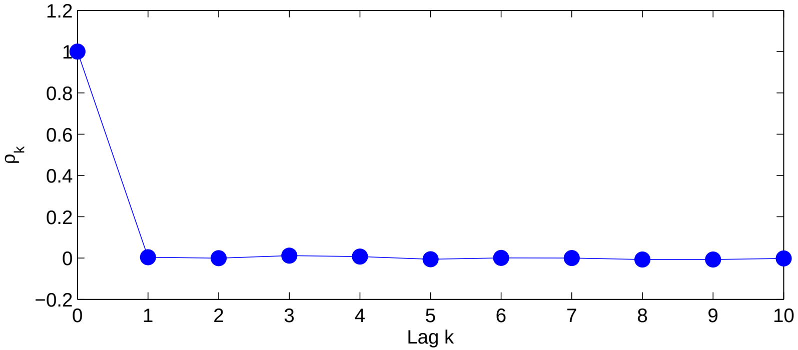
LIF $I = 15.7$:





Serial correlations of LIF

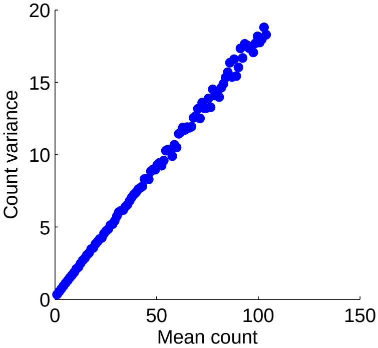
LIF $I = 15.7$:



Integrate-and-fire driven with white noise are still renewal processes!

Count statistics of LIF

LIF $I = 15.7$:

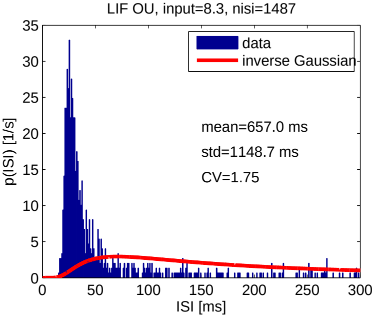


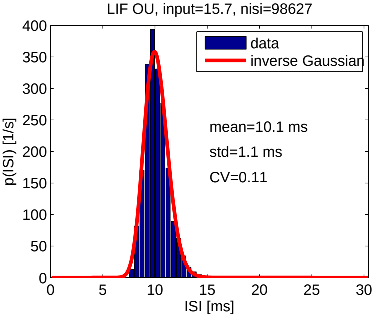
Fano factor is not one!

Interval statistics of LIF with OU noise

$$\tau \frac{dV}{dt} = -V + RI + U$$
$$\tau_{OU} \frac{dU}{dt} = -U + D\xi$$

Ohrnstein-Uhlenbeck noise is lowpass filtered white noise.

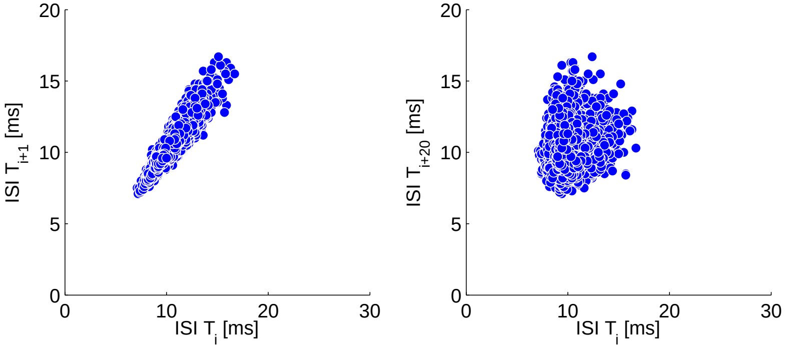




More peaky than the inverse Gaussian!

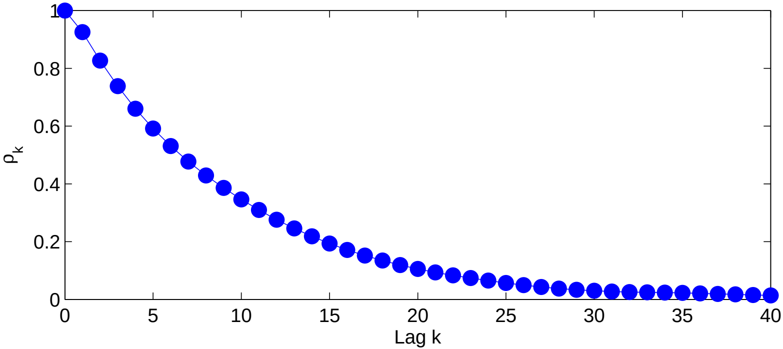
Interval return map of LIF with OU noise

LIF $I = 15.7$, $\tau_{OU} = 100\,\text{ms}$:



Serial correlations of LIF with OU noise

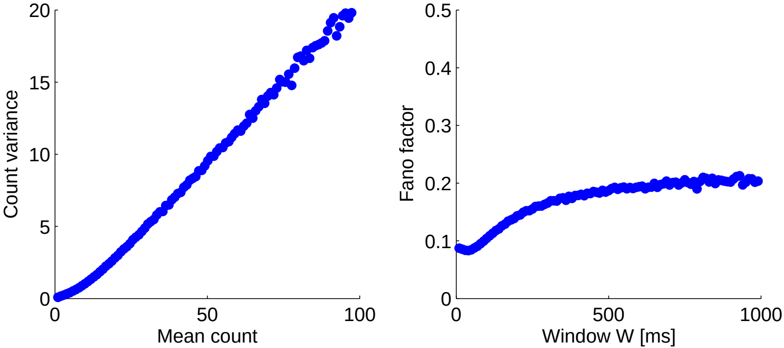
LIF $I = 15.7$, $\tau_{OU} = 100\,\mathrm{ms}$:



OU-noise introduces positive interval correlations!

Count statistics of LIF with OU noise

LIF $I = 15.7$, $\tau_{OU} = 100\,\mathrm{ms}$:

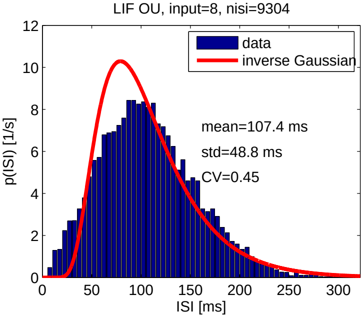


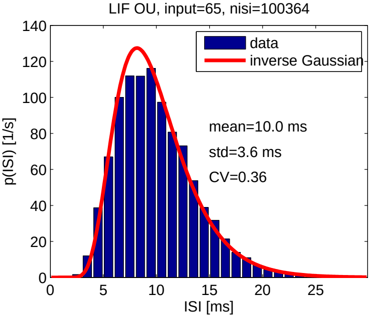
Fano factor increases with count window duration.

Interval statistics of LIF with adaptation

$$\tau \frac{dV}{dt} = -V - A + RI + D\xi$$
$$\tau_{adapt} \frac{dA}{dt} = -A$$

Adaptation A with time constant τ_{adapt} and increment ΔA at spike.

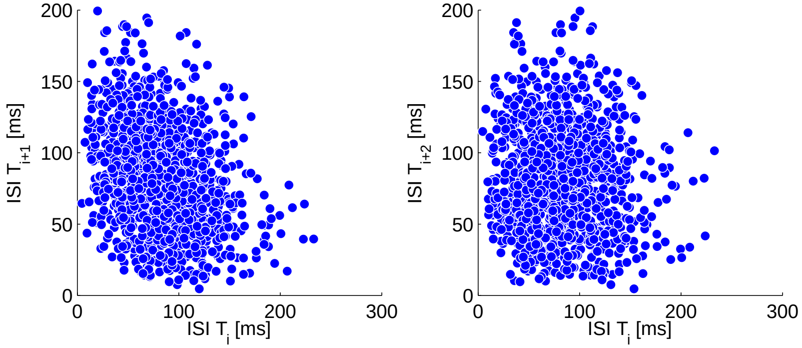




Similar to LIF with white noise.

Interval return map of LIF with adaptation

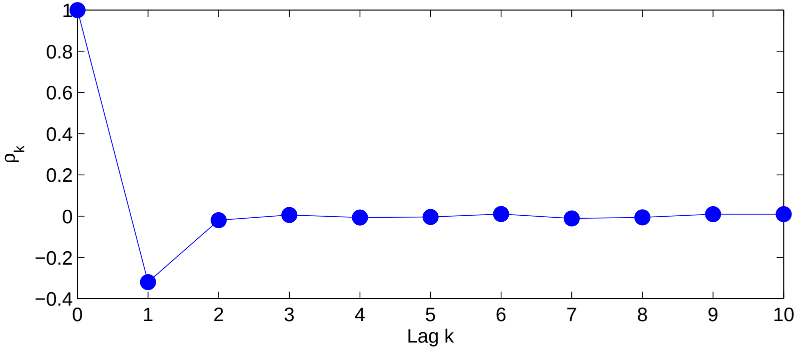
LIF $I = 10$, $\tau_{adapt} = 100\,\text{ms}$:



Negative correlation at lag one.

Serial correlations of LIF with adaptation

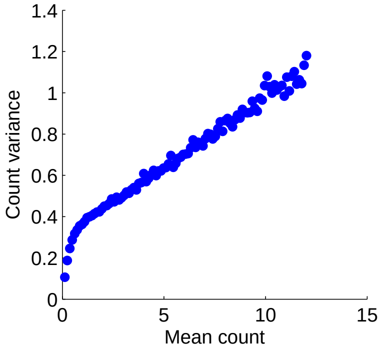
LIF $I = 10$, $\tau_{adapt} = 100\,\text{ms}$:

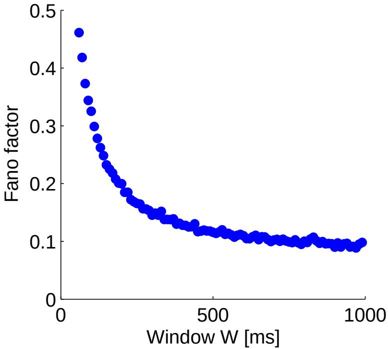


Adaptation with white noise introduces negative interval correlations!

Count statistics of LIF with adaptation

LIF $I = 10$, $\tau_{adapt} = 100\,\text{ms}$:





Fano factor decreases with count window duration.